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# $W^{r,p}(R)$-splines

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In [3] Golomb describes, for $1 < p < \infty$, the $H^{r,p}(R)$-extremal extension $F_*$ of a function $f: E \to R$ (i.e., the $H^{r,p}$-spline with knots in $E$) and studies the cone $H^{r,p}_{*E}$ of all such splines. We study the problem of determining when $F_*$ is in $W^{r,p} \equiv H^{r,p} \cap L^p$. If $F_* \in W^{r,p}$, then $F_*$ is called a *$W^{r,p}$-spline*, and we denote by $W^{r,p}_{*E}$ the cone of all such splines. If $E$ is quasiuniform, then $F_* \in W^{r,p}$ if and only if $\{f(t_i)\}_{t_i \in E} \in l^p$. The cone $W^{r,p}_{*E}$ with $E$ quasiuniform is shown to be homeomorphic to $l^p$. Similarly, $H^{r,p}_{*E}$ is homeomorphic to $h^{r,p}$. Approximation properties of the $W^{r,p}$-splines are studied and error bounds in terms of the mesh size $| E |$ are calculated. Restricting ourselves to the case $p = 2$ and to quasiuniform partitions $E$, the second integral relation is proved and better error bounds in terms of $| E |$ are derived.

## 1. INTRODUCTION

We denote by $H^{r,p}(R) \equiv H^{r,p}$, $1 < p < \infty$ and $r = 1, 2, \ldots$, the set of functions which have an absolutely continuous $(r - 1)$th derivative and whose $r$th derivative is in $L^p(R) \equiv L^p$. Let $E$ be a set of real numbers and $f$ a mapping of $E$ into the set of reals, R. Golomb [3] gives necessary and sufficient conditions for

$$\min_{\substack{x \in H^{r,p} \\ x|_E = f}} \int_R | D^r x |^p \tag{1.1}$$

to exist. This problem can be separated into two. In the first place, one must decide whether the set over which $x$ ranges is nonvoid, and secondly, if it is, it must be decided whether the resulting $L^p$ minimization problem has a solution. Solutions to (1.1) are called *extremal $H^{r,p}$-extensions* of $f$. If $E$ is a finite set consisting of more than $r - 1$ points, then $f$: $E \to R$ has a unique extremal $H^{r,p}$-extension $F_*$ which we call the *$H^{r,p}$-spline* interpolating $f$ on $E$. Golomb obtains necessary and sufficient conditions for (1.1) to exist in terms of the $H^{r,p}$-splines.

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THEOREM 1.1 (Golomb). *Suppose* $\{t_1, t_2, ...\}$ *is a dense subset of* $E$, $e_n = \{t_1, ..., t_n\}$, *and* $F_n$ *is the* $H^{r,p}$*-spline that interpolates* $f$ *on* $e_n$. *If* $f: E \to R$ *is continuous, then* $f$ *has an extremal* $H^{r,p}$*-extension if and only if the sequence*

$$\left\{\int_R | D^r F_n |^p\right\} \tag{1.2}$$

*is bounded.*

In this paper we will be interested in an extremal problem similar to (1.1). Let $W^{r,p} \equiv H^{r,p} \cap L^p$, so that $W^{r,p}$ is the usual Sobolev space of real-valued functions. Since $1 < p < \infty$, it is easy to see that $W^{r,p}$ is dense in $H^{r,p}$ when $H^{r,p}$ is supplied with the (uniformly rotund) norm

$$\| x \|_{H^{r,p}} = \left(\sum_{i=1}^{r} | x(t_i)|^p\right)^{1/p} + \| D^r x \|_{L^p}.$$

$W^{r,p}$ is a uniformly rotund Banach space when supplied with the norm $\| x \|_{W^{r,p}} = \| x \|_{L^p} + \| D^r x \|_{L^p}$. We call $x_*$ an *extremal* $W^{r,p}$*-extension* of $f: E \to R$ if

$$\min_{\substack{x \in W^{r,p} \\ x|_E = f}} \int_R | D^r x |^p. \tag{1.3}$$

is attained at $x = x_*$. We will, from time to time, use the term $W^{r,p}$-spline instead of "extremal $W^{r,p}$-extension." Necessary and sufficient conditions will be given for (1.3) to exist.

The set $W^{r,p}_{*E}$ of solutions to (1.3) is studied and various topological properties are derived. In particular it is shown that if $E$ is quasiuniform then $W^{r,p}_{*E}$ is homeomorphic to $l^p$, closed, and nowhere dense in $W^{r,p}$. Approximation properties of the $W^{r,p}$-splines are discussed in both the $L^p$ and $L^\infty$ norms. Interpolation space theory yields results in $L^q$ for $p < q < \infty$. When specializing to the Hilbert space case we find that for quasiuniform meshes the second integral relation holds and hence we can improve our error estimates for smooth functions in the usual manner. Again, interpolation space theory allows us to derive new error bounds for functions in the Besov spaces.

## 2. EXISTENCE AND CHARACTERIZATION

In order to simplify the statements of theorems and proofs we will henceforth assume that all the functions $f: E \to R$ have a continuous extension to $R$. In particular this means that we may assume $E$ is closed. We now indicate the relationship between the $W^{r,p}$ and $H^{r,p}$-extremal extension problem.

LEMMA 2.1. *Suppose $f\colon E \to R$. If $F_*$ is an extremal $W^{r,p}$-extension of $f$ then $F_*$ is an extremal $H^{r,p}$-extension of $f$.*

*Proof.* We set $V_f{}^r = \{D^rF\colon F \in W^{r,p}$ and $F|_E = f\}$. Clearly $V_f{}^r$ is a flat in $L^p$ and by hypothesis $D^rF_* \in V_f{}^r$ is the element of smallest $L^p$ norm in $V_f{}^r$. It follows from [8, p. 18], that

$$\int_R (|D^rF_*|^{p-1} \operatorname{sgn} D^rF_*)\, D^rG = 0 \tag{2.1}$$

for all $G \in W^{r,p}$ which vanish on $E$. If we restrict our attention to functions $G$ which are infinitely differentiable with compact support we conclude (in a completely analogous manner to that of [3, Theorem 4.2a]):

(i) $|D^rF_*|^{p-1} \operatorname{sgn} D^rF_* \in C^r(R\backslash E)$,

(ii) $D^r(|D^rF_*(t)|^{p-1} \operatorname{sgn} D^rF_*(t)) = 0$, $t \in R\backslash E$,

(iii) $D^rF_*(t) = 0$ for $t < \inf E$ and $t > \sup E$,

(iv) $D^k(|D^rF_*|^{p-1} \operatorname{sgn} D^rF_*)$ exists and is continuous for $t \in R\backslash E'$ where $E'$ is the set of limit points of $E$ and $k = 0, 1,\ldots, r-2$. (2.2)

In Section 5 of [3] it is shown that these conditions characterize all solutions of the $H^{r,p}$-extremal problem, the solution being unique if the cardinality of $E$ is greater than or equal to $r$. Thus, it follows that $F_*$ is a solution to the $H^{r,p}$-extremal extension problem and this completes the proof.

It is easy to see that the $H^{r,p}$-extremal extension problem has a solution if and only if the set $\{x\colon x|_E = f$ and $x \in H^{r,p}\}$ is not empty. The situation for the $W^{r,p}$-extremal extension problem is not so simple. The set $\{x\colon x|_E = f$ and $x \in W^{r,p}\}$ may well be nonempty and still (1.3) may have no solution. To illustrate this fact we develop the following necessary condition for a solution to (1.3).

COROLLARY 2.1. *Suppose $F_*$ is an extremal $W^{r,p}$-extension of $f\colon E \to R$, then*

$$\operatorname{supp} F_* \subset \operatorname{co}(E), \tag{2.3}$$

*where* $\operatorname{co}(E)$ *denotes the convex hull of $E$.*

This is an obvious consequence of (2.2 iii). In particular for finite point sets $E = \{t_1,\ldots, t_n\}$ and $f\colon E \to R$, if $f(t_i) \neq 0$ for $i = 1,\ldots, n$ then there are many $W^{r,p}$-extensions of $f$ but no $W^{r,p}$-extremal extension.

We now come to a $W^{r,p}$ analog of Theorem 1.1.

THEOREM 2.1. *Suppose $f\colon E \to R$, $\{t_1, t_2,\ldots\}$ is a dense subset of $E$, and $F_n$ is the $H^{r,p}$-spline which interpolates $f$ on $e_n = \{t_1,\ldots, t_n\}$. Then $f$ has an* *extremal $W^{r,p}$-extension if and only if there exist intervals $I_n$ so that $I_n \subset I_{n+1}$, $\bigcup_{n=1}^{\infty} I_n = R$, and*

$$\sup_{1 \leqslant n < \infty} \left( \int_{I_n} |F_n|^p + |D^r F_n|^p \right) < \infty. \tag{2.4}$$

*Proof.* Suppose $f$ has an extremal $W^{r,p}$-extension $F_*$. Lemma 2.1 implies that $F_*$ is also an extremal $H^{r,p}$-extension of $f$. Theorem 1.1 then tells us that

$$\sup_{1 \leqslant n < \infty} \int_R |D^r F_n|^p < \infty. \tag{2.5}$$

Furthermore in [3, Theorem 2.2], we find that $F_n$ converges to $F_*$ in the normed space $H^{r,p}$. Since convergence in $H^{r,p}$ implies uniform convergence on compact sets, there is an integer $N_k$ so that for all $n \geqslant N_k$

$$\int_{-k}^{k} |F_n|^p \leqslant 2 \int_R |F_*|^p, \qquad k = 0, 1, \ldots . \tag{2.6}$$

We may assume that the sequence $\{N_k\}$ is strictly increasing. If we set

$$I_n = [-k, k], \qquad N_k \leqslant n < N_{k+1}, \tag{2.7}$$

we see that $I_n \subset I_{n+1}$ and $\bigcup_{n=1}^{\infty} I_n = R$. Now (2.5) and (2.6) imply (2.4) with the $I_n$ chosen as in (2.7).

Conversely, if (2.4) holds then we set

$$G_n(t) = \left\{ \begin{array}{ll} F_n(t), & t \in I_n = [\alpha_n, \beta_n] \\ \displaystyle\sum_{k=0}^{r-1} D^k F_n(\alpha_n) \frac{(t-\alpha_n)^k}{k!}, & t \in (-\infty, \alpha_n) \\ \displaystyle\sum_{k=0}^{r-1} D^k F_n(\beta_n) \frac{(t-\beta_n)^k}{k!}, & t \in (\beta_n, \infty) \end{array} \right\}. \tag{2.8}$$

By (2.4), $\{G_n\}$ is a bounded sequence in $H^{r,p}$. Since $H^{r,p}$ is reflexive, a subsequence, say $\{G_{n_k}\}$, converges weakly to $G_*$. Now $G_* |_E = f$ because point evaluations are in $(H^{r,p})^*$, the continuous dual of $H^{r,p}$. Thus, $G_*$ is an $H^{r,p}$-extension of $f$, and it follows that $f$ has an extremal $H^{r,p}$-extension $F_*$. As before, Theorem 2.2 of [3] implies that $F_n$ converges locally uniformly to $F_*$. Consider the sequence $\{X_{I_n} \cdot F_n\}_{n=1}^{\infty}$ of $L^p$ functions. By (2.4) this sequence is bounded in $L^p$ and hence a subsequence converges weakly to (say) $B_* \in L^p$. Since $F_n$ converges locally uniformly to $F_*$ we must have $B_* = F_*$ (a.e.). Thus, $F_* \in L^p$ and since $F_* \in H^{r,p}$ also, then $F_* \in W^{r,p}$. Clearly, $F_*$ is the extremal $W^{r,p}$-extension of $f$ and this completes the proof.

## 3. SPECIAL CASES

In this section we consider extending functions $f: E \to R$ to $W^{r,p}$, where $\text{co}(E) = R$ and $E = \{t_i\}_{i=-\infty}^{\infty}$, $t_i < t_{i+1}$ for $i = 0, \pm 1, \pm 2, \ldots$. Particularly, we will be interested in necessary and sufficient conditions for the extremal $W^{r,p}$-extension to exist. To this end we set

$$A_{r,p} = \{g \in H^{r,p}[0, 1]: g(0) = 1$$

and

$$D^r(| D^r g(t)|^{p-1} \operatorname{sgn} D^r g(t)) = 0 \text{ for } t \in [0, 1]\}. \tag{3.1}$$

If $F_*$ is an $H^{r,p}$-spline with no knots in $(0, 1)$ and $F_*(0) = 1$, there is a $g_* \in A_{r,p}$ so that $F_* |_{[0,1]} = g_*$. We let

$$\eta_{r,p} = \inf_{g \in A_{r,p}} \int_0^1 | g |^p. \tag{3.2}$$

LEMMA 3.1. *$\eta_{r,p} > 0$ for $1 < p < \infty$ and $r = 1, 2, \ldots$.*

*Proof.* We let $P_r$ be the set of polynomials of degree $r - 1$ or less. Clearly, $g \in A_{r,p}$ if and only if

$$\begin{aligned}
&\text{(i)} \quad g(t) = Q(t) + \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{r-1}} | P(\tau_r)|^{1/(p-1)} \\
&\qquad\qquad \cdot \operatorname{sgn}(P(\tau_r))\, d\tau_r \cdots d\tau_1, \\
&\text{(ii)} \quad g(0) = 1, \\
&\text{(iii)} \quad Q \in P_r \text{ and } P \in P_r.
\end{aligned} \tag{3.3}$$

Now using the local compactness of $P_r$ and the continuous dependence of $g$ on $Q$ and $P$, $Q(0) = 1$, the result quickly follows.

From Lemma 3.1 we obtain certain necessary conditions, in terms of the interpolation conditions and the mesh, that a $W^{r,p}$-spline exist.

COROLLARY 3.1. *Let $E = \{t_i\}_{i=-\infty}^{\infty}$, $t_i < t_{i+1}$, and let $f$ map $E$ into $R$. If an extremal $W^{r,p}$-extension exists, then*

$$\begin{aligned}
&\text{(i)} \quad \sum_{i=-\infty}^{\infty} | f(t_i)|^p \,| t_i - t_{i+1} | < \infty, \\
&\text{(ii)} \quad \sum_{i=-\infty}^{\infty} | f(t_i)|^p \,| t_{i-1} - t_i | < \infty.
\end{aligned} \tag{3.4}$$

*Proof.* We let $F_*$ be the extremal $W^{r,p}$-extension of $f$. For each interval $(t_i, t_{i+1})$ we have

$$\int_{t_i}^{t_{i+1}} | F_*(t)|^p \, dt = \int_0^1 | F_*(\tau[t_{i+1} - t_i] + t_i)|^p (t_{i+1} - t_i) \, d\tau$$
$$\geqslant \eta_{r,p} | F_*(t_i)|^p (t_{i+1} - t_i). \tag{3.5}$$

The last inequality follows from (2.2 ii) and the definition of $\eta_{r,p}$. Since $\eta_{r,p}$ is positive by Lemma 3.1 and $F_* \in L^p$, we may sum the end terms of (3.5) over $i$ to obtain

$$\infty > (\eta_{r,p}^{-1}) \int_R | F_*(t)|^p$$
$$\geqslant \sum_{i=-\infty}^{\infty} | f(t_i)|^p \, | t_{i+1} - t_i |. \tag{3.6}$$

This implies (3.4 i) and (3.4 ii) follows in a similar manner.

Let us review the definition of the extremal $W^{r,p}$-extension in order to point out a sufficient condition for a solution to exist. We are given an $f: E \to R$ and if we set $V_f = \{x \in W^{r,p}: x |_E = f\}$ then (1.3) becomes

$$\inf_{x \in V_f} \int_R | D^r x |^p. \tag{3.7}$$

We define $V_f^r = \{D^r x: x \in V_f\}$, so that $V_f^r$ is a flat in $L^p$, and the solution $F_*$ to (3.7), if it exists, has an $r$th derivative which is the best approximant to $\theta$ (the zero element in $L^p$) from $V_f^r$. Thus $F_*$ exists if

$$\inf_{x \in V_f^r} \int_R | x |^p \tag{3.8}$$

is attained. $L^p$ is uniformly rotund so that (3.8) has a (unique) solution if $V_f^r$ is closed. Assuming that $V_f^r$ is not empty, it is easy to see that $V_f^r$ is closed if and only if $V_0^r = \{D^r x: x \in W^{r,p}$ and $x |_E = 0\}$ is closed. It follows that one way to prove the existence of extremal $W^{r,p}$-extensions is to show

(i) $V_f \neq \varnothing$,

(ii) $V_0^r$ is closed in $L^p$. (3.9)

We now further specialize our partition $E$. The set $E = \{t_i\}_{i=-\infty}^{\infty}$ is called *quasiuniform* if there are two strictly positive numbers $\delta_1$ and $\delta_2$ so that

$$\delta_1 \leqslant (t_{i+1} - t_i) \leqslant \delta_2, \qquad i = 0, \quad \pm 1, \quad \pm 2, \ldots . \tag{3.10}$$

LEMMA 3.2. *If $E = \{t_i\}_{i=-\infty}^{\infty}$ is quasiuniform, then $V_0^r$ is closed in $L^p$.*

*Proof.* Suppose $\{g_n\}_{n=1}^{\infty} \subset V_0{}^r$ and $g_n$ converges to $g$ in $L^p$. Since $g_n \in V_0{}^r$, there are $G_n \in W^{r,p}$ so that $G_n|_E = 0$ and $D^r G_n = g_n$. Clearly $G_n$ converges in $H^{r,p}$ to $G$ where (due to the local uniform convergence)

$$\text{(i)} \quad G|_E = 0,$$
$$\text{(ii)} \quad D^r G = g. \tag{3.11}$$

Thus, all we need to show is that $G$ is in $L^p$. Let $\delta_1$ and $\delta_2$ be the constants of quasiuniformity for $E$ as in (3.10). Using Rolle's theorem repeatedly one can see that there are quasiuniform sequences $\{u_j{}^k\}_{j=-\infty}^{\infty}$, $k = 0, 1,..., r-1$, satisfying

$$\text{(i)} \quad D^k G(u_j{}^k) = 0, \; j = 0, \pm 1, \pm 2,...; \; k = 0, 1,..., r-1,$$
$$\text{(ii)} \quad \delta_1 \leqslant u_{j+1}^k - u_j^k \leqslant 3^{k+1}\delta_2, \qquad j = 0, \pm 1, \pm 2,...;$$
$$k = 0, 1,..., r-1. \tag{3.12}$$

We define inductively a function similar to the greatest integer function $[\cdot]$ by

$$\text{(i)} \quad [t]_0 = \{u_j{}^0 : u_j{}^0 \leqslant t \leqslant u_{j+1}^0\},$$
$$\text{(ii)} \quad [t]_k = \{u_j{}^k : u_j{}^k \leqslant [t]_{k-1} < u_{j+1}^k\}, \qquad k = 1,..., r-1. \tag{3.13}$$

It is easy to see that

$$G(t) = \int_{[t]_0}^{t} \int_{[t]_1}^{\tau_1} \cdots \int_{[t]_{r-1}}^{\tau_{r-1}} g(\tau_r)\, d\tau_r \cdots d\tau_1. \tag{3.14}$$

Finally, we note that $G \in L^p$ because

$$\begin{aligned}
\int_R |G|^p &= \sum_{i=-\infty}^{\infty} \int_{u_i^0}^{u_{i+1}^0} |G|^p \\
&\leqslant \sum_{i=-\infty}^{\infty} \int_{u_i^0}^{u_{i+1}^0} \left(\int_{[t]_0}^{t} \cdots \int_{[t]_{r-1}}^{\tau_{r-1}} |g(\tau_r)|\, d\tau_r \cdots d\tau_1\right)^p dt \\
&\leqslant \sum_{i=-\infty}^{\infty} \int_{u_i^0}^{u_{i+1}^0} \left(\int_{[u_i^0]_0}^{u_{i+1}^0} \cdots \int_{[u_i^0]_{r-1}}^{u_{i+1}^0} |g(\tau_r)|\, d\tau_r \cdots d\tau_1\right)^p dt \\
&\leqslant \sum_{i=-\infty}^{\infty} (u_{i+1}^0 - u_i{}^0) \left(\prod_{k=0}^{r-2} (u_{i+1}^0 - [u_i{}^0]_k)\right)^p (u_{i+1}^0 - [u_i{}^0]_{r-1})^{p/q} \\
&\quad \cdot \int_{[u_i^0]_{r-1}}^{u_{i+1}^0} |g|^p, \qquad 1/p + 1/q = 1.
\end{aligned} \tag{3.15}$$

Using (3.12) and (3.13) we note that

$$(u_{i+1}^0 - [u_i^0]_k) \leqslant \delta_2 \left(\sum_{j=1}^{k+1} 3^j\right) = \delta_2 \left(\frac{3^{k+2} - 3}{2}\right), \qquad k = 0, 1,..., r-1. \tag{3.16}$$

It follows that

$$\begin{aligned}\int_R |G|^p &\leqslant K \sum_{i=-\infty}^{\infty} \int_{[u_i^0]_{r-1}}^{u_{i+1}^0} |g|^p \\ &\leqslant K \sum_{i=-\infty}^{\infty} \int_{[u_i^0]_{r-1}}^{[u_i^0]_{r-1}+\delta_2((3^{r+1}-3)/2)} |g|^p \\ &\leqslant K \left(\frac{\delta_2}{\delta_1}\left(\frac{3^{r+1}-3}{2}\right)\right) \sum_{i=-\infty}^{\infty} \int_{u_i^{r-1}}^{u_i^{r-1}+\delta_2((3^{r+1}-3)/2)} |g|^p \\ &\leqslant K \left(\frac{\delta_2}{\delta_1}\left(\frac{3^{r+1}-3}{2}\right)\right)^2 \int_R |g|^p. \end{aligned} \tag{3.17}$$

The constant $((\delta_2/\delta_1)((3^{r+1} - 3)/2))^2$ in the last inequality results from the fact that we are integrating over possibly overlapping intervals. By (3.16) we may take

$$K = \delta_2^{pr}(3) \left(\prod_{k=0}^{r-2} \left\{\frac{3^{k+2} - 3}{2}\right\}\right)^p \left(\frac{3^{r+1} - 3}{2}\right)^{p/q}. \tag{3.18}$$

Since $g \in L^p$ we have $G \in L^p$ also. Therefore, $V_0^r$ is closed since $\{g_n\}_{n=1}^{\infty}$ was an arbitrary Cauchy sequence. As a corollary we observe

COROLLARY 3.2. *If $E \subset R$ contains a quasiuniform sequence then $V_0^r$ is closed.*

From Corollary 3.2 we draw the following corollary.

COROLLARY 3.3. *Suppose $f$: $E \to R$, with $E$ containing a quasiuniform sequence. Then $f$ has a $W^{r,p}$-extension if and only if it has an extremal $W^{r,p}$-extension.*

*Proof.* If $f$ has a $W^{r,p}$-extension then Corollary 3.2 implies that (3.9) holds, and, hence, $f$ has an extremal $W^{r,p}$-extension. The converse is clear. We note that this corollary is in contradistinction to the case when $E$ is finite and there are many $W^{r,p}$-extensions and possibly no extremal extensions.

We are now in a position to prove one of the main results of this paper.

THEOREM 3.1. *Let $E = \{t_i\}_{i=-\infty}^{\infty}$, $t_{i+1} > t_i$, be quasiuniform, $1 < p < \infty$, and $f$ map $E$ into $R$. There is an extremal $W^{r,p}$-extension of $f$ if and only if $\{f(t_i)\}_{i=-\infty}^{\infty}$ is in $l^p$.*

*Proof.* If there is an extremal $W^{r,p}$-extension, Corollary 3.1 implies

$$\sum_{i=-\infty}^{\infty} |f(t_i)|^p \, | t_{i+1} - t_i | < \infty. \tag{3.19}$$

Since $E$ is quasiuniform this means that $\{f(t_i)\}_{i=-\infty}^{\infty}$ is in $l^p$.

Conversely, suppose that $\{f(t_i)\}_{i=-\infty}^{\infty} \in l^p$ and $E$ is quasiuniform with $0 < \delta_1 \leqslant t_{i+1} - t_i \leqslant \delta_2$. Let $\phi$ be an infinitely differentiable real-valued function whose support is contained in $(-\delta_1/4, \delta_1/4)$, and set $\phi(0) = 1$. It is then easy to verify that

$$\sum_{i=-\infty}^{\infty} f(t_i)\, \phi(t - t_i) \in W^{r,p}. \tag{3.20}$$

Since $E$ is quasiuniform, Lemma 3.2 tells us that $V_0{}^r$ is closed. Thus, (3.9) holds and the extremal $W^{r,p}$-extension problem has a solution.

We obtain from Theorem 3.1 a corollary which yields insight into the structure of $W^{r,p}$.

COROLLARY 3.4. *If $F \in W^{r,p}$ and $E = \{t_i\}_{i=-\infty}^{\infty}$ is quasiuniform then $\{D^k F(t_i)\}_{i=-\infty}^{\infty} \in l^p$ for $k = 0, 1,..., r-1$.*

*Proof.* Since $F \in W^{r,p}$ and $E$ is quasiuniform, Corollary 3.3 implies that $F|_E$ has an extremal $W^{r,p}$-extension. Theorem 3.1 then tells us that $\{F(t_i)\}_{i=-\infty}^{\infty} \in l^p$. Now $D^k F \in W^{r-k,p}$ for $k = 0,..., r-1$, and by applying the same argument we get $\{D^k F(t_i)\}_{i=-\infty}^{\infty} \in l^p$.

In closing this section we would like to point out that Theorem 3.1 tells us that we have an extremal extension (hence an extension) of a function under certain circumstances if it is in $l^p$. Similar results for $H^{r,p}$-extensions have been derived by Golomb [3], with the difference being that instead of requiring $f$ to be in $l^p$ one requires that the $r$th divided difference of $f$ to be in $l^p$. Jerome and Schumaker [5] and Schoenberg [6] treat the problem of determining whether a function $x$ is in $H^{r,p}$. They get necessary and sufficient conditions for $x$ to be in $H^{r,p}$ in terms of $r$th divided differences of $x$ being uniformly bounded in some weighted $l^p$ spaces.

## 4. The $W^{r,p}$-Spline Operators

Suppose $E \subset R$; we define

$$\text{(i)}\quad W_{*E}^{r,p} = \{x \in W^{r,p}\colon x \text{ is the extremal } W^{r,p}\text{-extension of } x\,|_E\},$$
$$\text{(ii)}\quad H_{*E}^{r,p} = \{x \in H^{r,p}\colon x \text{ is the extremal } H^{r,p}\text{-extension of } x\,|_E\}. \tag{4.1}$$

We will feel free to drop the superscripts $(r, p)$ when no ambiguity results. The set $W_{*E}^{r,p}(H_{*E}^{r,p})$ is called the set of $W^{r,p}$-splines ($H^{r,p}$-splines) with knots in $E$. Throughout this section we retain the assumption that $E$ is closed.

The approximation map, $T_E^{r,p}$, assigns to a $W^{r,p}$ function $x$ the $W^{r,p}$-spline, $T_E^{r,p}x$, which is the extremal $W^{r,p}$-extension of $x\,|_E$ (when such an extension exists). Again we will drop the superscripts $(r, p)$ when possible. The approximation map, $S_E^{r,p}$, which maps $H^{r,p}$ onto the set $H_{*E}$ is defined in a similar manner.

If the cardinality of $E$ is larger than or equal to $r$, then $S_E^{r,p}$ is well defined. Conditions that make $T_E$ well defined are stated in the following theorem.

Theorem 4.1. *The map $T_E$ is well defined if and only if $E$ contains a quasiuniform sequence.*

*Proof.* If $E$ contains a quasiuniform sequence then Corollary 3.3 implies that $T_E$ is well defined.

If $\operatorname{co}(E) = R$ and $E$ does not contain a quasiuniform sequence, then without loss of generality we may assume that there are two sequences $\{\alpha_j^k\}_{j=0}^{\infty}$, $k = 1, 2$, of real numbers satisfying

$$\begin{aligned}
&\text{(i)}\quad \alpha_j^k \in E,\ j = 0, 1,\ldots;\ k = 1, 2,\\
&\text{(ii)}\quad \alpha_j^1 < \alpha_j^2,\ j = 0, 1,\ldots,\\
&\text{(iii)}\quad E \cap (\alpha_j^1, \alpha_j^2) = \varnothing,\ j = 0, 1,\ldots,\\
&\text{(iv)}\quad (\alpha_j^2 - \alpha_j^1) \to \infty \text{ as } j \to \infty.
\end{aligned} \tag{4.2}$$

There is a $\{\gamma_j\}_{j=-\infty}^{\infty} \in l^p$ so that the sum $\sum_{j=0}^{\infty} |\gamma_j|^p(\alpha_j^2 - \alpha_j^1)$ diverges. It is easy to see, following the construction of (3.20), that there is a function $x \in W^{r,p}$ so that $x(\alpha_j^2) = \gamma_j$ for $j = 0, 1,\ldots$. Corollary 3.1 then implies that $x\,|_E$ has no extremal $W^{r,p}$-extension, and, hence, $T_E$ is not defined at $x$.

Finally, if $\operatorname{co}(E) \neq R$ we may assume that $\inf(E) = t_0 > -\infty$. Clearly there is a $y \in W^{r,p}$ so that $y(t_0) \neq 0$. Corollary 2.1 tells us that $y$ can have no extremal $W^{r,p}$-extension. This completes the proof of Theorem 4.1.

Theorem 4.2. *If $E$ contains a quasiuniform sequence the map $T_E$ is hemilinear, idempotent, continuous, and bounded.*

*Proof.* We set $V_0 = \{x \in W^{r,p}: x\,|_E = 0\}$. Clearly $V_0$ is closed in $W^{r,p}$ and, by the assumption on $E$, $V_0{}^r$ is closed in $L^p$ (Corollary 3.2). The map

$$D^r: V_0 \to V_0{}^r \tag{4.3}$$

is continuous, linear, and onto. By the open mapping theorem $D^r\,|_{V_0}$ has a continuous inverse which we denote by $D^{-r}$. We set $P_{V_0{}^r} = P$ to be the (continuous) metric projection from $L^p$ onto $V_0{}^r$. Thus we may write

$$T_E = I - D^{-r} \circ P \circ D^r, \tag{4.4}$$

where $I$ is the identity map from $W^{r,p}$ to $W^{r,p}$. Theorem 4.2 follows by examining the right side of (4.4).

For the map $S_E$ we have similarly the following theorem.

THEOREM 4.3. *If $E$ contains $r$ or more points then $S_E^{r,p}$ is hemilinear, idempotent, continuous, and bounded.*

This is a strengthening of Theorem 6.2 of [3] where Golomb proves that $S_E$ is slightly continuous. Theorem 4.3 is a special case of a more general result of Holmes [4].

We now assume that $E = \{t_i\}_{i=-\infty}^{\infty}$ is a quasiuniform partition with $0 < \delta_1 < t_{i+1} - t_i < \delta_2$ for all integers $i$. For any sequence $\gamma = \{\gamma_i\}_{i=-\infty}^{\infty}$ we define

$$f_\gamma : E \to R \qquad \text{by} \quad f_\gamma(t_i) = \gamma_i\,. \tag{4.5}$$

Let $l^p$ denote the Banach space of doubly infinite sequences $\gamma$ which have finite norm, $\|\gamma\|_{l^p} = (\sum_{i=-\infty}^{\infty} |\gamma_i|^p)^{1/p}$. Similarly, let $h^{r,p}$ be the space of sequences $\gamma$ for which the norm

$$\|\gamma\|_{h^{r,p}} = \left(\sum_{i=1}^{r} |\gamma_i|^p + \sum_{i=-\infty}^{\infty} |f_\gamma(t_i,\ldots, t_{i+r})|^p\right)^{1/p} \tag{4.6}$$

is finite. The notation $f_\gamma(t_i,\ldots, t_{i+r})$ means the $r$th divided difference of $f_\gamma$ on the point set $\{t_i,\ldots, t_{i+r}\}$. From (3.20) we see that there is a linear injection $I_E : l^p \to W^{r,p}$ defined by $I_E(\gamma) = \sum_{i=-\infty}^{\infty} f_\gamma(t_i)\,\phi(t - t_i)$. It is not hard to see that $I_E$ is continuous. We define a map, $t_E^{r,p}$, with domain $l^p$ by

$$t_E^{r,p} = T_E^{r,p} \circ I_E\,. \tag{4.7}$$

The superscripts again may be dropped when no confusion arises.

THEOREM 4.4. *The map $t_E$ is a homeomorphism which takes Cauchy sequences into Cauchy sequences between $l^p$ with its norm topology and $W_{*E}$ with its relative $W^{r,p}$ norm topology.*

*Proof.* The map $t_E$ is continuous because it is the composition of two continuous maps.

In order to see that $(t_E)^{-1}$ is continuous we first prove a result of independent interest.

LEMMA 4.1. *The $W^{r,p}$ norm $\| x \|_{L^p} + \| D^r x \|_{L^p}$ is equivalent to the norm*

$$| x |_{W^{r,p}} = \Big( \sum_{i=-\infty}^{\infty} \sum_{k=0}^{r-1} | D^k x(t_i)|^p \Big)^{1/p} + \| D^r x \|_{L^p}, \tag{4.8}$$

*where it is assumed that $E = \{t_i\}_{i=-\infty}^{\infty}$ is quasiuniform.*

*Proof.* For any $x \in W^{r,p}$ we have

$$\begin{aligned} \| x(t) \|_{L^p} &= \Big\| \sum_{i=-\infty}^{\infty} \chi_{[t_i, t_{i+1})}(t) \Big( \sum_{k=0}^{r-1} \frac{D^k x(t_i)}{k!} (t - t_i)^k \\ &\quad + \int_{t_i}^{t} \cdots \int_{t_i}^{\tau_{r-1}} D^r x(\tau_r)\, d\tau_r \cdots d\tau_1 \Big) \Big\|_{L^p} \\ &\geqslant \Big\| \sum_{i=-\infty}^{\infty} \chi_{[t_i, t_{i+1})}(t) \Big( \sum_{k=0}^{r-1} \frac{D^k x(t_i)}{k!} (t - t_i)^k \Big) \Big\|_{L^p} \\ &\quad - \Big\| \sum_{i=-\infty}^{\infty} \chi_{[t_i, t_{i+1})}(t) \int_{t_i}^{t} \cdots \int_{t_i}^{\tau_{r-1}} D^r x(\tau_r)\, d\tau_r \cdots d\tau_1 \Big\|_{L^p} \\ &= I_1 - I_2 . \end{aligned} \tag{4.9}$$

Using the facts that norms are equivalent on finite dimensional spaces and $E$ is quasiuniform, we find that there is a positive constant $C$ depending only on $E$ so that

$$I_1 \leqslant C \Big( \sum_{i=-\infty}^{\infty} \sum_{k=0}^{r-1} | D^k x(t_i)|^p \Big)^{1/p}. \tag{4.10}$$

By an argument similar to that in (3.15) we see that

$$I_2 \leqslant \delta_2^r \| D^r x \|_{L^p} . \tag{4.11}$$

Therefore,

$$(1 + C + \delta_2^r) \| x \|_{W^{r,p}} \geqslant C | x |_{W^{r,p}} . \tag{4.12}$$

Convergence in $| \cdot |_{W^{r,p}}$ in particular implies convergence in $H^{r,p}$ and furthermore that the limit is an $H^{r,p}$ function which interpolates $l^p$ data on a quasi-uniform partition. It follows that $W^{r,p}$ is complete in $| \cdot |_{W^{r,p}}$ since $x \in H^{r,p}$

and $\{x(t_i)\}_{i=-\infty}^{\infty} \in l^p$ imply that $x \in W^{r,p}$. This can be seen by subtracting any $W^{r,p}$ function $y$ which interpolates $x$ on $E$ from $x$ and noting that the proof of Lemma 3.2 tells us that $x - y \in L^p$. We apply the open mapping theorem to get a constant $K$ so that $| x |_{W^{r,p}} \geqslant K \| x \|_{W^{r,p}}$. This completes the proof. An immediate corollary follows.

COROLLARY 4.1. *The norm* $\| \cdot \|_{W^{r,p}}$ *is equivalent to*

$$| x |'_{W^{r,p}} = \|\{x(t_i)\}\|_{l^p} + \| D^r x \|_{L^p} .$$

We now return to the proof of Theorem 4.4. Let $x_n \in W_{*E}$ and $x_n$ converge to $x_* \in W_{*E}$, the convergence being in the $W^{r,p}$ norm topology and hence in $| \cdot |'_{W^{r,p}}$. Corollary 4.1 then implies that

$$(t_E)^{-1}(x_n) = \{x_n(t_i)\} \xrightarrow{l^p} \{x(t_i)\}. \tag{4.13}$$

Thus $t_E$ is a homeomorphism and clearly $(t_E)^{-1}$ takes Cauchy sequences into Cauchy sequences, obviously so does $t_E$, and thus Theorem 4.4 is proved.

If the proof of Theorem 3.1 of [3] is examined one sees that there is a continuous linear map $L$: $h^{r,p} \to H^{r,p}$ so that $L(\gamma)|_E = f_\gamma$. We define a map, $s_E^{r,p}$, analogous to $t_E$ by

$$s_E^{r,p} \equiv S_E^{r,p} \circ L. \tag{4.14}$$

THEOREM 4.5. *The map* $s_E^{r,p}$ *is a homeomorphism between* $h^{r,p}$ *and* $H_{*E}^{r,p}$.

*Proof.* By [3], Theorem 6.3, we know that $(s_E)^{-1}$ is continuous. To prove the forward continuity we note that $s_E$ is the composition of two continuous maps. For other results on homeomorphic images of spline sets we refer the reader to [4].

## 5. TAYLOR FIELDS

As in Section 4 we assume here that $E = \{t_i\}_{i=-\infty}^{\infty}$, $t_{i+1} > t_i$, is quasi-uniform. Let $\mu$ be a map from $E$ into the set of integers $\{1,..., r\}$. Suppose for each $t_i \in E$ we have defined numbers $f_0(t_i),..., f_{\mu(t_i)-1}(t_i)$. We list the elements of $E$ in order, repeated according to the multiplicity function $\mu$, starting at $t_0$. Thus we obtain the doubly infinite sequence

$$\cdots \leqslant \tau_{-1} < \tau_0 \leqslant \tau_1 \leqslant \cdots, \tag{5.1}$$

where $t_0 = \tau_0 = \tau_1 = \tau_{\mu(t_0)-1} < \tau_{\mu(t_0)}$. On this new sequence we define a real-valued function $f$ by $f(\tau_k) \equiv f_j(t_i)$, where $\tau_k = t_i$ and $\tau_{k-j} = t_i$ but

$\tau_{k-j-1} = t_{i-1}$. We call $x_*$ a $W^{r,p}$-spline with knots of multiplicity $\mu(t_i)$ at $t_i$ interpolating $f$ if

$$\min_{x \in W^{r,p}} \left\{ \int_R |D^r x|^p : D^k x(t_i) = (k!) f_k(t_i), \qquad i = 0, \pm 1, \pm 2, \ldots, \quad k = 0, 1, \ldots, \mu(t_i) - 1 \right\} \tag{5.2}$$

is attained at $x = x_*$. Since $W^{r,p}$ is complete in $|\cdot|_{W^{r,p}}$, it is easy to see that (5.2) has a solution. Similarly, we define an $H^{r,p}$-spline with knots of multiplicity $\mu(t_i)$ at $t_i$ interpolating $f$ by replacing $W^{r,p}$ in (5.2) by $H^{r,p}$. Set

$W^{r,p}_{*E,\mu}$ = set of solutions to (5.2) as $f$ varies over all functions from $\{\tau_i\}_{i=-\infty}^{\infty}$ into $R$. (5.3)

The set $H^{r,p}_{*E,\mu}$ is defined analogously. Let $\gamma = \{\gamma_i\}_{i=-\infty}^{\infty}$ be a sequence of real numbers. We define $f_\gamma : \{\tau_i\}_{i=-\infty}^{\infty} \to R$ by $f_\gamma(\tau_i) = \gamma_i$. Subsets of all sequences $\gamma$ may be normed by

$$\begin{aligned} &\text{(i)} \quad \|\gamma\|_{l^p} = \Big( \sum_{i=-\infty}^{\infty} |f_\gamma(\tau_i)|^p \Big)^{1/p}, \\ &\text{(ii)} \quad \|\gamma\|_{h^{r,p}} = \Big( \sum_{i=0}^{r-1} |f_\gamma(\tau_i)|^p + \sum_{i=-\infty}^{\infty} |f_\gamma(\tau_i, \ldots, \tau_{i+r})|^p \Big)^{1/p}, \end{aligned} \tag{5.4}$$

where the extended divided difference in (ii) is defined in the natural way (see e.g. [3]). The set of sequences for which (i) is finite is called $l^p$ and the set for which (ii) is finite is called $h^{r,p}$. As before, we define the map $t^{r,p}_{E,\mu}$ which maps $l^p$ onto $W^{r,p}_{*E,\mu}$ by assigning to $\gamma \in l^p$ the $W^{r,p}$-spline with knots of multiplicity $\mu(t_i)$ at $t_i$ interpolating $f_\gamma$. Similarly we define $s^{r,p}_{E,\mu}$ which maps $h^{r,p}$ onto $H^{r,p}_{*E,\mu}$ by assigning to $\gamma \in h^{r,p}$ the $H^{r,p}$-spline with knots of multiplicity $\mu(t_i)$ at $t_i$ interpolating $f_\gamma$.

We now state two theorems without proof since the proofs are easy generalizations of proofs in the previous sections. (For the $W^{r,p}$ case take special note of Lemma 4.1 and Corollary 4.1.)

THEOREM 5.1. *The map $t^{r,p}_{E,\mu}$ is a homeomorphism between $l^p$ and $W^{r,p}_{*E,\mu}$ (both with the usual norm topology). Moreover, both $t^{r,p}_{E,\mu}$ and its inverse take Cauchy sequences into Cauchy sequences.*

THEOREM 5.2. *The map $s^{r,p}_{E,\mu}$ is a homeomorphism between $h^{r,p}$ and $H^{r,p}_{*E,\mu}$.*

It is easy to see that both $W^{r,p}_{*E,\mu}$ and $H^{r,p}_{*E,\mu}$ are cones. In [3] Golomb proves that $H^{r,p}_{*E,\mu}$ is closed and nowhere dense in $H^{r,p}$.

THEOREM 5.3. *Suppose $E$ is quasiuniform. The cone $W^{r,p}_{*E,\mu}$ is closed and nowhere dense in $W^{r,p}$.*

*Proof.* $W^{r,p}_{*E,\mu}$ is closed since the homeomorphism $(t^{r,p}_{E,\mu})^{-1}$ by Theorem 5.1 takes Cauchy sequences into Cauchy sequences. The proof of the nowhere density is essentially the same as in [3], Theorem 6.1, and will be omitted.

## 6. APPROXIMATION BY $W^{r,p}$-SPLINES

In this section $E$ will be any set of real numbers so that $\mathrm{co}(E) = R$ and $| E |$, the maximum spacing between elements of $E$, is bounded. In particular, this means that $E$ contains a quasiuniform sequence and hence $T_E$ is well defined. Given any $x \in W^{r,p}$ we want to measure

$$\| D^j(x - T_E^{r,p}x)\|_{L^p}, \qquad j = 0, 1,..., r, \tag{6.1}$$

in terms of $| E |$ and $\| x \|_{W^{r,p}}$. We state the fundamental lemma.

LEMMA 6.1. *Suppose $x \in W^{r,p}$ and $x |_E = 0$. Then there is a constant $K^r$, depending only on $r$ and $p$, so that*

$$\| x \|_{L^p} \leqslant K^r | E |^r \| D^r x \|_{L^p}. \tag{6.2}$$

*Proof.* Let $x$ and $E$ be as above. In a process similar to that of (3.12) we can find sequences $\{u_j^k\}_{j=-\infty}^{\infty}$, $k = 0, 1,..., r - 1$, satisfying

$$\begin{aligned} &\text{(i)} \quad D^k x(u_j^k) = 0, \qquad j = 0, \pm 1, \pm 2,...; \qquad k = 0, 1,..., r-1, \\ &\text{(ii)} \quad | E | \leqslant u_{j+1}^k - u_j^k \leqslant 3^{k+1} | E |, \qquad j = 0, \pm 1, \pm 2,...; \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 0, 1,..., r-1. \end{aligned} \tag{6.3}$$

We write

$$x(t) = \int_{[t]_0}^{t} \cdots \int_{[t]_{r-1}}^{\tau_{r-1}} D^r x(\tau_r)\, d\tau_r \cdots d\tau_1 \tag{6.4}$$

and taking into account (3.15)–(3.18) we obtain

$$\begin{aligned} \| x \|_{L^p} &\leqslant | E |^r \left( \prod_{k=0}^{r-1} \left( \frac{3^{k+2} - 3}{2} \right) \right) \left( \frac{3^{r+2} - 3^2}{2} \right)^{1/p} \| D^r x \|_{L^p} \\ &= K^r | E |^r \| D^r x \|_{L^p}. \end{aligned} \tag{6.5}$$

This is (6.2). Using this lemma we can now prove the following theorem.

THEOREM 6.1. *There are constants $K_j^r$, depending only on r, p, and j so that for every $x \in W^{r,p}$ and $j = 0, 1,\dots, r$*

$$\| D^j(x - T_E^{r,p}x)\|_{L^p} \leqslant K_j^r \mid E \mid^{r-j} \| D^r x \|_{L^p}$$
$$\leqslant K_j^r \mid E \mid^{r-j} \| x \|_{W^{r,p}} . \tag{6.6}$$

*Proof.* For $j = 0, 1,\dots, r-1$ we note, using Rolle's theorem, that there is a quasiuniform sequence $E_j$ satisfying $\mid E_j \mid \leqslant (j+1) \mid E \mid$ and $D^j(x - T_E^{r,p}x)(t) = 0$ for $t \in E_j$. Since $D^j(x - T_E^{r,p}x) \in W^{r-j,p}$ for $j = 0, 1,\dots, r-1$, we may apply Lemma 6.1 with $r$ replaced by $r - j$ obtaining

$$\| D^j(x - T_E^{r,p}x)\|_{L^p} \leqslant K^{r-j} \mid E_j \mid^{r-j} \| D^r(x - T_E^{r,p}x)\|_{L^p}$$
$$\leqslant K^{r-j}(j+1)^{r-j} \mid E \mid^{r-j} \| D^r(x - T_E^{r,p}x)\|_{L^p} . \tag{6.7}$$

From representation (4.4) we observe that

$$\| D^r(x - T_E^{r,p}x)\|_{L^p} = \| P \circ D^r x \|_{L^p} \leqslant 2 \| D^r x \|_{L^p} , \tag{6.8}$$

since metric projections are always bounded by twice the norm of their argument. Thus, setting $K_j^r = 2K^{r-j}(j+1)^{r-j}$ for $j = 0, 1,\dots, r-1$ and $K_r^r = 2$, (6.7) and (6.8) yield the first half of (6.6) and the second half is clear.

The derivation of $L^\infty$-error bounds is even easier.

LEMMA 6.2. *Suppose $x \in W^{r,p}$ and $x \mid_E = 0$. Then there is a constant $K^r$, depending only on r and p, so that*

$$\| x \|_{L^\infty} \leqslant K^r \mid E \mid^{r-1/p} \| D^r x \|_{L^p} , \qquad j = 0, 1,\dots, r-1. \tag{6.9}$$

*Proof.* There are sequences $\{u_j^k\}_{j=-\infty}^{\infty}$, $k = 0, 1,\dots, r-1$, satisfying (6.3) for $x$. Furthermore, by representation (6.4) and Hölders inequality we obtain

$$| x(t)| \leqslant \Big(\prod_{k=0}^{r-2} (t - [t]_k)\Big) (t - [t]_{r-1})^{1/q} \Big(\int_{[t]_{r-1}}^t \mid D^r x(\tau_r)|^p \, d\tau_r\Big)^{1/p}$$
$$\leqslant \mid E \mid^{r-1+1/q} \Big(\prod_{k=0}^{r-2} \Big(\frac{3^{k+2}-3}{2}\Big)\Big)\Big(\frac{3^{r+1}-3}{2}\Big)^{1/q} \| D^r x \|_{L^p}$$
$$\leqslant K^r \mid E \mid^{r-1/p} \| D^r x \|_{L^p} . \tag{6.10}$$

The last inequality in (6.10) is independent of $t$ so that by taking the supremum of the end terms in (6.10) yields (6.9).

THEOREM 6.2. *There are constants $K_j{}^r$, depending on r, p, and j only, so that for every $x \in W^{r,p}$ we have*

$$\begin{aligned} \| D^j(x - T_E^{r,p}x)\|_{L^\infty} &\leqslant K_j{}^r \mid E \mid^{r-j-1/p} \| D^r x \|_{L^p} \\ &\leqslant K_j{}^r \mid E \mid^{r-j-1/p} \| x \|_{W^{r,p}}, \qquad j = 0, 1,..., r-1. \end{aligned} \tag{6.11}$$

The course of the proof of Theorem (6.2) is essentially the same as that of Theorem (6.1) and can safely be omitted. We also obtain the following corollary.

COROLLARY 6.1. *There are constants $K_j{}^r$, depending only on r, p, and j, so that for every $x \in H^{r,p}$ and $j = 0, 1,..., r-1$,*

$$\| D^j(x - S_E^{r,p}x)\|_{L^\infty} \leqslant K_j{}^r \mid E \mid^{r-j-1/p} \| D^r x \|_{L^p} . \tag{6.12}$$

Let us now consider the Hilbert space ($W^{r,2}$) case in greater detail. We assume that $E = \{t_i\}_{i=-\infty}^{\infty}$, $t_{i+1} > t_i$, is quasiuniform. From the characterization (2.2) of the $W^{r,2}$-splines we find that for $x \in W^{r,2}$, $T_E^{r,2}x$ is piecewise polynomial of degree $2r - 1$ and $T_E^{r,2}(x) \in C^{2r-2}(R)$. These are familiar properties of splines. If we were working on a finite interval $I$ we would conclude that the spline is in $W^{2r-1,2}(I)$. In this direction we have the following lemma.

LEMMA 6.3. *If E is quasiuniform and $x \in W^{r,2}$ then $T_E^{r,2}x \in W^{2r-1,2}$.*

*Proof.* We write

$$T_E^{r,2}x = \sum_{i=-\infty}^{\infty} \chi_{[t_i, t_{i+1})}(t) \left( \sum_{j=0}^{2r-1} \frac{D^j x(t_i{+})}{j!} (t - t_i)^j \right). \tag{6.13}$$

$T_E^{r,2}x$ is in $L^2$ so by using the quasiuniformity and the equivalence of norms on finite-dimensional spaces we obtain

$$\| T_E^{r,2}x \|_2 \geqslant C \left( \sum_{i=-\infty}^{\infty} \sum_{j=0}^{2r-1} | D^j x(t_i{+})|^2 \right)^{1/2}, \qquad C > 0. \tag{6.14}$$

In particular $\{D^{2r-1}x(t_i +)\}_{i=-\infty}^{\infty} \in l^2$, and it follows that $T_E^{r,2}x \in W^{2r-1,2}$ since it possesses the required continuity conditions. We are now in a position to prove the *second integral relation.*

LEMMA 6.4. *If E is quasiuniform then for all $x \in W^{2r,2}$ we have*

$$\int_R \{D^r(x - T_E^{r,2}x)\}^2 = \int_R (-1)^r (D^{2r}x)(x - T_E^{r,2}x). \tag{6.15}$$

*Proof.* The proof of this result follows the standard integration by parts argument. We calculate ($E = \{t_i\}_{i=-\infty}^{\infty}$, $t_{i+1} > t_i$)

$$\begin{aligned}
\int_R \{D^r(x - T_E^{r,2}x)\}^2 &= \lim_{N\to\infty} \int_{t_{-N}}^{t_N} \{D^r(x - T_E^{r,2}x)\}^2 \\
&= \lim_{N\to\infty} \Big\{ \int_{t_{-N}}^{t_N} (-1)^r (D^{2r}x)(x - T_E^{r,2}x) \\
&\quad + \sum_{\substack{|k|\leqslant N \\ k\neq N}} \sum_{j=0}^{r-1} (-1)^j\, D^{r+j}(x - T_E^{r,2}x) \\
&\quad \cdot D^{r-j-1}(x - T_E^{r,2}x) \,\Big|_{t_k}^{k+1} \Big\} \\
&= (-1)^r \int_R (D^{2r}x)(x - T_E^{r,2}x) \\
&\quad + \lim_{N\to\infty} \sum_{j=0}^{r-2} (-1)^j\, D^{r+j}(x - T_E^{r,2}x) \\
&\quad \cdot D^{r-j-1}(x - T_E^{r,2}x) \,\Big|_{t_{-N}}^{t_N}. \qquad (6.16)
\end{aligned}$$

In order to reach the last equality we have used the fact that $x$ and $T_E^{r,2}x$ are in $C^{2r-2}(R)$ and that $(x - T_E^{r,2}x)|_E = 0$. Since $(x - T_E^{r,2}x) \in W^{2r-1,2}$, Corollary (3.4) implies $\{D^j(x - T_E^{r,2}x)(t_i)\}_{i=-\infty}^{\infty} \in l^2$ for $j = 0, 1,..., 2r - 2$. Thus the limit in the last term of (6.16) tends to zero as $N \to \infty$. This yields (6.15) which is also known as the second integral relation. Once the second integral relation is proved, it is well known (see [7]) that better error estimates can be derived for smooth functions.

THEOREM 6.3. *Suppose $E$ is quasiuniform; then there are constants $M_j^r$ and $N_j^r$, depending only on $j$ and $r$, so that for all $x \in W^{2r,2}$*

$$\begin{aligned}
&\text{(i)} \quad \| D^j(x - T_E^{r,2}x)\|_{L^2} \leqslant M_j^r \,|\, E \,|^{2r-j} \| D^{2r}x \|_{L^2} \\
&\qquad \leqslant M_j^r \,|\, E \,|^{2r-j} \| x \|_{W^{2r,2}}, \qquad j = 0,..., r, \\
&\hspace{25em} (6.17) \\
&\text{(ii)} \quad \| D^j(x - T_E^{r,2}x)\|_{L^\infty} \leqslant N_j^r \,|\, E \,|^{2r-j-1/2} \| D^{2r}x \|_{L^2} \\
&\qquad \leqslant N_j^r \,|\, E \,|^{2r-j-1/2} \| x \|_{W^{2r,2}}, \quad j = 0, 1,..., r - 1.
\end{aligned}$$

*Proof.* We will just sketch the proof since the ideas are not new. Applying the Cauchy–Schwarz inequality to the right side of (6.15) and then using estimate (6.7) with $j = 0$ yields

$$\| D^r(x - T_E^{r,2}x)\|_{L^2}^2 \leqslant \| D^{2r}x \|_{L^2} K^r \mid E \mid^r \| D^r(x - T_E^{r,2}x)\|_{L^2}. \quad (6.18)$$

Dividing both sides by $\| D^r(x - T_E^{r,2}x)\|_{L^2}$ and again using (6.7) we get

$$\begin{aligned} \| D^j(x - T_E^{r,2}x)\|_{L^2} &\leqslant \mid E \mid^{r-j} K^{r-j}(j + 1)^{r-j} \| D^r(x - T_E^{r,2}x)\|_{L^2} \\ &\leqslant M_j{}^r \mid E \mid^{2r-j} \| D^{2r}x \|_{L^2}. \end{aligned} \quad (6.19)$$

The second half of (6.17(i)) is clear and (6.17(ii)) follows in a similar manner.

## 7. Interpolation Space Results

Butzer and Berens [2] describe methods of obtaining intermediate Banach spaces given two Banach spaces $X_1$ and $X_2$. Using the $K$-method for generating intermediate Banach spaces $(X_1, X_2)_{\theta,q}$ $(0 < \theta < 1, 1 \leqslant q < \infty$ and/or $0 \leqslant \theta \leqslant 1, q = \infty)$ permits us to state [2, p. 180].

Theorem 7.1. *Let $X_i$, $Y_i$, $i = 1, 2$, be Banach spaces and $T$ be a linear mapping from $X_1 + X_2$ into $Y_1 + Y_2$ so that $T|_{X_i}: X_i \to Y_i$, $i = 1, 2$. If $\| T|_{X_i} \| = M_i$, $i = 1, 2$, then*

(i) $T: (X_1, X_2)_{\theta,q} \to (Y_1, Y_2)_{\theta,q}$,

(ii) $$\| T \|_{\theta,q} = \sup_{1=\|x\|_{(X_1,X_2)_{\theta,q}}} \| Tx \|_{(Y_1,Y_2)_{\theta,q}} \leqslant M_1^{1-\theta} M_2{}^{\theta}. \quad (7.1)$$

We have developed error bounds for $W^{r,p}$-spline approximation in both the $L^p$ and $L^\infty$ norms. In the last two sections we obtained bounds for $W^{r,2}$ and $W^{2r,2}$ functions. Theorem 7.1 will then yield more error bounds in intermediate spaces. We note that $(L^p, L^\infty)_{\theta,q}$ is equal to $L^q$ when $1/q = (1 - \theta)/p$ with equivalent norms, see [2] Theorem 3.3.6, and that $(X, X)_{\theta,q} = X$.

Theorem 7.2. *There are constants $C_j$ depending only on $1 < p \leqslant q \leqslant \infty$ and $j = 0, 1, \ldots, r - 1$ so that for every $x \in W^{r,p}$ we have*

$$\| D^j(x - T_E^{r,p}x)\|_{L^q} \leqslant C_j \mid E \mid^{r-j+(1/q-1/p)} \| x \|_{W^{r,p}}. \quad (7.2)$$

*Proof.* We set $X_1 = X_2 = W^{r,p}$, $Y_1 = L^p$ and $Y_2 = L^\infty$. Suppose

$f \in (W^{r,p})^*$, the continuous dual of $W^{r,p}$, with norm 1. The map $T^j_{x_0}$ defined on $W^{r,p}$ by

$$T^j_{x_0}(x) = \{D^j(x_0 - T_E^{r,p}x_0)\} \cdot f(x) \tag{7.3}$$

is linear, continuous, and by (6.6) and (6.11) satisfies

$$\text{(i)} \quad \sup_{1=\|x\|_{W^{r,p}}} \| T^j_{x_0}(x)\|_{L^p} \leqslant K_j^r \mid E \mid^{r-j} \| x_0 \|_{W^{r,p}},$$

$$\text{(ii)} \quad \sup_{1=\|x\|_{W^{r,p}}} \| T^j_{x_0}(x)\|_{L^\infty} \leqslant K_j^{*r} \mid E \mid^{r-j-1/p} \| x_0 \|_{W^{r,p}}, \tag{7.4}$$

for $j = 0, 1,..., r - 1$. Thus, $T^j_{x_0}$ fulfills the requirements of Theorem 7.1, and since

$$(r - j)(1 - \theta) + (r - j - 1/p)\theta = (r - j) - \theta/p = (r - j) + (1/q - 1/p)$$

we obtain (7.2).

The $W^{r,2}$-spline operator $T_E^{r,2}$ is linear so that we may apply Theorem 7.1 directly. The development here closely follows that of Varga [9]. It is known that $(W^{r,2}, W^{2r,2})_{\theta,q} = B_2^{\sigma,q}$, $\sigma = (1 + \theta)r$. The spaces $B_2^{\sigma,2}$ are called Besov spaces [1]. Furthermore, if $\sigma$ is an integer and $q = 2$ then $B_2^{\sigma,q} = W^{\sigma,2}$. The maps $x \mapsto D^j(x - T_E^{r,2}x)$, $j = 0, 1,..., r - 1$, take $W^{r,2}$ and $W^{2r,2}$ linearly into both $L^2$ and $L^\infty$ with the bounds (6.6), (6.11), and (6.17). Theorem 7.1 yields the following theorem.

THEOREM 7.3. *If $x \in B_2^{\sigma,q}$, $\sigma = r(1 + \theta)$, and $q = 2/(1 - \theta)$ then*

$$\| D^j(x - T_E^{r,2}x)\|_{L^q} \leqslant M_j \mid E \mid^{\sigma - j + 1/q - 1/2} \| x \|_{B_2^{\sigma,q}}, \; j = 0, 1,..., r - 1. \tag{7.5}$$

Finally, setting $q = 2$ and keeping the range space $L^2$ we obtain the following theorem.

THEOREM 7.4. *If $n$ is an integer between $r$ and $2r$ and $x \in W^{n,2}$, then for $j = 0, 1,..., r$ we have*

$$\| D^j(x - T_E^{r,2}x)\|_{L^2} \leqslant M \mid E \mid^{n-j} \| x \|_{W^{n,2}}. \tag{7.6}$$

The proofs of both these theorems follow immediately from Theorem 7.1 and the error bounds previously derived.

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